Qualifying Exam – Partial Differential Equations (II) – August 2019

INSTRUCTION: *Any resources or communications are NOT allowed during the exam. You may use all well-known results without proof unless you are asked to prove such a result, but you must indicate the result and include necessary steps. Work on familiar problems first.*

#1. Let Ω be a bounded open set in $\mathbf{R}^n$. Show that there exist constants $c_1, c_2 > 0$ such that

$$c_1\|u\|_{H^2(\Omega)} \le \|\Delta u\|_{L^2(\Omega)} \le c_2\|u\|_{H^2(\Omega)} \quad \forall\, u \in H_0^2(\Omega).$$

Furthermore, if Ω is a bounded domain with $\partial\Omega \in C^2$, show that the same result holds for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$. (**Henceforth, a domain means a connected open set**.)

#2. Let $n \ge 2$ and $1 < p < n$ be given. Show that the inequality

$$\int_{\mathbf{R}^n} \frac{|u|^q}{|x|^r}dx \le C\int_{\mathbf{R}^n} |Du|^p dx \quad \forall\, u \in W_0^{1,p}(\mathbf{R}^n),$$

where $C > 0$ is a constant, can only hold for certain q and r. Find such q and r, and **prove the inequality**.

#3. Let Ω be a bounded domain in $\mathbf{R}^n$ with $\partial\Omega \in C^1$ and $\alpha > 0$. Show that there exists a constant C depending only on n and α such that the inequality

$$\|u\|_{L^2(\Omega)} \le C\|Du\|_{L^2(\Omega)}$$

holds for all $u \in H^1(\Omega)$ with $|\{x \in \Omega \mid u(x) = 0\}| \ge \alpha$.

#4. Let Ω be a bounded domain in $\mathbf{R}^n$ with $\partial\Omega \in C^1$ and let $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$. Define reasonably a weak solution $u \in H^1(\Omega)$ to boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u + \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega \end{cases}$$

so that any $C^2(\bar\Omega)$ solution is a weak solution. Prove the existence and uniqueness of a weak solution to the problem.

#5. Let $\Omega = \{x \in \mathbf{R}^n : a < |x| < b\}$, where $b > a \ge 0$ and $n \ge 2$, and let $S\colon \mathbf{R} \to \mathbf{R}$ be continuous. Assume $U \in H_0^1(a, b)$ is a weak solution of

$$-(U'(r)r^{n-1})' = S(U(r))r^{n-1} \quad \text{in } (a, b).$$

Define $u(x) = U(|x|)$. Show that u belongs to $H_0^1(\Omega) \cap C^{0,\frac{1}{2}}(\bar\Omega)$ and is a weak solution of

$$-\Delta u = S(u) \quad \text{in } \Omega.$$

#6. Let Ω be a bounded open set in $\mathbf{R}^n$, a_{ij}, b_i, $c \in L^\infty(\Omega)$, and assume the operator

$$Lu = -\sum_{i,j=1}^n D_i(a_{ij}(x)D_j u) + \sum_{i=1}^n b_i(x)D_i u + c(x)u$$

is uniformly elliptic on Ω. Show that the BVP $\begin{cases} Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$ has a unique weak solution $u \in H_0^1(\Omega)$ for each $f \in L^2(\Omega)$ if either (a) or (b) given below holds:

(a) $c(x) \ge 0$ a.e. in Ω. (b) $b_i \in W^{1,\infty}(\Omega)$, $c(x) \ge \frac{1}{2}\sum_{i=1}^n D_i b_i(x)$ a.e. in Ω.